ON THE ACCUMULATION OF DISTURBANCES IN TRANSIENT LINEAR IMPULSIVE SYSTEMS

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Theories of impulsive systems have in recent years formed the subject of an extensive literature, including work by Gardner and Burns [1], Bulgakov [2], Tsypkin [3], Ragazzini and Zadeh [4], and a number of other authors.

The present paper studies the problem of the accumulation of disturbances in transient linear impulsive systems under the influence of external forces of bounded magnitudes.

Consider the impulsive system, described by the difference equations

$$y_k(t+\tau) + \sum_{l=1}^{n} a_{kl}(t)\, y_l(t) = x_k(t) \qquad (k = 1,\ldots,n) \tag{1}$$

The system of scalar equations (1) is equivalent to the matrix equation

$$y(t+\tau) + a(t)\, y(t) = x(t) \tag{2}$$

where $y(t)$, $a(t)$, $x(t)$ are the matrices

$$y(t) = \| y_k(t) \|, \qquad a(t) = \| a_{kl}(t) \|, \qquad x(t) = \| x_k(t) \| \tag{3}$$

Denoting by $\Theta(t)$ the square matrix whose columns are the linearly independent solutions of the homogeneous matrix equation

$$y(t+\tau) + a(t)\, y(t) = 0 \tag{4}$$

the solution of the equation (2) may be represented in the form

$$y(t) = \Theta(t)\,\Theta^{-1}(t-\vartheta\tau)\, y^*(t-\vartheta\tau) + \sum_{j=1}^{\vartheta} \Theta(t)\,\Theta^{-1}(t-\vartheta\tau+j\tau)\, x(t-\vartheta\tau+j\tau-\tau) \tag{5}$$

In this formula, the quantity θ denotes the integral part of t/τ and the matrix $\Theta^{-1}(t)$ is the inverse of the matrix $\Theta(t)$. Since the second term in (5) vanishes in the interval $0 < t < \tau$, one has by (5)

$$y(t) = y^*(t) \qquad (0 < t < \tau) \tag{6}$$

i.e. the unknown matrix $y(t)$ coincides at any point of the interval $0 < t < \tau$ with the matrix $y^*(t)$, given in advance.

By (5), the elements of the matrix $y(t)$ will have the form

$$y_s(t) = \sum_{k=1}^{n} [\Theta(t)\Theta^{-1}(t-\vartheta\tau]_{sk}\, y_k^*(t-\vartheta\tau) + $$
$$+ \sum_{k=1}^{n}\sum_{j=1}^{\vartheta} [\Theta(t)\Theta^{-1}(t-\vartheta\tau+j\tau)]_{sk}\, x_k(t-\vartheta\tau+j\tau-\tau) \quad (s = 1,\ldots,n) \tag{7}$$

Denoting by $N(t, j\tau)$ the matrix weight function

$$N(t, j\tau) = \Theta(t)\Theta^{-1}(t-\vartheta\tau+j\tau) \tag{8}$$

the solution (5) may be written

$$y(t) = N(t, 0)\, y^*(t-\vartheta\tau) + \sum_{j=1}^{\vartheta} N(t, j\tau)\, x(t-\vartheta\tau+j\tau-\tau) \tag{9}$$

Correspondingly, the expressions (7) take the form

$$y_s(t) = \sum_{k=1}^{n} N_{sk}(t, 0)\, y_k^*(t-\vartheta\tau) + \sum_{k=1}^{n}\sum_{j=1}^{\vartheta} N_{sk}(t, j\tau)\, x_k(t-\vartheta\tau+j\tau-\tau) \quad (s = 1,\ldots,n) \tag{10}$$

The state of the system at the fixed instant of time $t = t_1$ is determined by the quantities (10) for $t = t_1$; for this purpose, the quantity θ must be replaced by θ_1, i.e. by the integral part of t_1/τ.

Assuming now that the external forces $x_k(t)$ are of bounded magnitude

$$|x_k(t)| \leqslant L_k \tag{11}$$

one obtains from (10) for $t = t_1$ an estimate of the largest possible displacement of some coordinate y_s at the instant of time $t = t_1$:

$$|y_s(t_1)| \leqslant \left| \sum_{k=1}^{n} N_{sk}(t_1, 0)\, y_k^*\,(t_1-\vartheta_1\tau) \right| + \sum_{k=1}^{n} L_k \sum_{j=1}^{\vartheta_1} |N_{sk}(t_1, j\tau)| \tag{12}$$

The expression (12) also gives a solution of the problem to hand. However, it will be noted that, as it is well known, the determination of the fundamental matrix $\Theta(t)$, and consequently also of the matrix weight function $N(t, j\tau)$ represents a difficult problem and that it can effectively be achieved for only special types of the coefficient matrix $a(t)$. Therefore interest attaches to the statement of an algorithm which permits the deduction of the estimate (12) with the help of a numerical scheme. For this purpose, consider the system of difference equations

$$Y_k(t) + \sum_{l=1}^{n} a_{lk}(t) Y_l(t+\tau) = 0 \qquad (k = 1, \ldots, n) \tag{13}$$

The coefficient matrix of the equations (13) is the transposed matrix of the matrix $a(t)$ of the original system (1). The system of difference equations (13) is conjugate to the original system (1). Multiplying the k-th equation ($k = 1, \ldots, n$) of the system (1) by $Y_k(t + \tau)$ and the μ-th equation ($\mu = 1, \ldots, n$) of the system (13) by $-y_\mu(t)$ and adding the left and right-hand sides respectively of all the equations thus obtained, one finds

$$\Delta \sum_{k=1}^{n} Y_k(t)\, y_k(t) = \sum_{k=1}^{n} Y_k(t+\tau) x_k(t) \tag{14}$$

Hence

$$\sum_{k=1}^{n} Y_k(t)\, y_k(t) = \sum_{k=1}^{n} Y_k^*(t - \vartheta\tau)\, y_k^*(t - \vartheta\tau) + \\ + \sum_{k=1}^{n} \sum_{j=1}^{\vartheta} Y_k(t - \vartheta\tau + j\tau)\, x_k(t - \vartheta\tau + j\tau - \tau) \tag{15}$$

Using (6), it is readily seen that the $Y_k^*(t)$ entering into (15) are functions, given in advance, which coincide in the interval $0 < t < \tau$ with the functions $Y_k(t)$.

For the instant $t = t_1$, the relationship (15) takes the form

$$\sum_{k=1}^{n} Y_k(t_1)\, y_k(t_1) = \sum_{k=1}^{n} Y_k(t_1 - \vartheta_1\tau)\, y_k^*(t_1 - \vartheta_1\tau) + \\ + \sum_{k=1}^{n} \sum_{j=1}^{\vartheta_1} Y_k(t_1 - \vartheta_1\tau + j\tau)\, x_k(t_1 - \vartheta_1\tau + j\tau - \tau) \tag{16}$$

Taking into consideration that $0 < t_1 - \theta_1\tau < \tau$, the asterisk on $Y_k^*(t_1 - \theta_1\tau)$ in (16) may be omitted. For the determination of the solution of the system of difference equations (13) one has to know in the interval $0 < t < \tau$ the function $Y_k^*(t)$, which in this interval, coincides with the unknown function $Y_k(t)$.

It will now be demanded that the unknown function $Y_k(t)$ satisfies at any instant of time t in the interval $\theta_1\tau < t < (\theta_1 + 1)\tau$ the condition

$$Y_s(t) = 1, \qquad Y_l(t) = 0 \qquad (l = 1, \ldots, s-1, s+1, \ldots, n) \tag{17}$$

Obviously, the conditions (17) together with the equations (13) also already uniquely determine the functions $Y_k^*(t)$. Subjected to the condition (17), the expression (15) takes the form

$$y_s(t_1) = \sum_{k=1}^{n} Y_k(t_1 - \vartheta_1\tau)\, y_k^*(t_1 - \vartheta_1\tau) + \sum_{k=1}^{n}\sum_{j=1}^{\vartheta_1} Y_k(t_1 - \vartheta_1\tau + j\tau)\, x_k(t_1 - \vartheta_1\tau + j\tau - \tau) \tag{18}$$

The expression (12) which is of interest here and which determines for $|x_k(t)| < L_k$ the largest possible deflection of some coordinate y_s at the instant of time $t = t_1$ now takes the form

$$|y_s(t_1)| \leqslant \left| \sum_{k=1}^{n} Y_k(t_1 - \vartheta_1\tau)\, y_k^*(t_1 - \vartheta_1\tau) \right| + \sum_{k=1}^{n} L_k \sum_{j=1}^{\vartheta_1} |Y_k(t_1 - \vartheta_1\tau + j\tau)| \tag{19}$$

where $Y_k(t)$ is the solution of the system of difference equations (13), satisfying the conditions (17). Likewise, from the above, there follows a method for the determination of the quantity $|y_s(t_1)|$ with the help of an electronic computer.

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